

CAT(0) GROUPS AND COXETER GROUPS WHOSE BOUNDARIES ARE SCRAMBLED SETS

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ABSTRACT. In this paper, we study CAT(0) groups and Coxeter groups whose boundaries are scrambled sets. Suppose that a group G acts geometrically (i.e. properly and cocompactly by isometries) on a CAT(0) space X . (Such group G is called a *CAT(0) group*.) Then the group G acts by homeomorphisms on the boundary ∂X of X and we can define a metric $d_{\partial X}$ on the boundary ∂X . The boundary ∂X is called a *scrambled set* if for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$, (1) $\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0$ and (2) $\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0$. We investigate when are boundaries of CAT(0) groups (and Coxeter groups) scrambled sets.

1. INTRODUCTION

The purpose of this paper is to study CAT(0) groups and Coxeter groups whose boundaries are scrambled sets.

Definitions and basic properties of CAT(0) spaces and their boundaries are found in [2]. A *geometric* action on a CAT(0) space is an action by isometries which is proper ([2, p.131]) and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space ([2, p.132]). A group G is called a *CAT(0) group*, if there exists a geometric action of G on some CAT(0) space X . Here we say that the boundary ∂X of X is a *boundary of G* . We note that if G is hyperbolic then the group G determines the boundary ∂X . In general, a CAT(0) group G does not determine the boundary ∂X of a CAT(0) space X on which G acts geometrically (Croke and Kleiner [5]).

Suppose that a group G acts geometrically on a CAT(0) space X . Then the group G acts by homeomorphisms on the boundary ∂X of X . We note that if $|\partial X| > 2$ then the boundary ∂X is uncountable.

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We define a metric on the boundary ∂X as follows: We first fix a basepoint $x_0 \in X$. Let $\alpha, \beta \in \partial X$. There exist unique geodesic rays $\xi_{x_0, \alpha}$ and $\xi_{x_0, \beta}$ in X with $\xi_{x_0, \alpha}(0) = \xi_{x_0, \beta}(0) = x_0$, $\xi_{x_0, \alpha}(\infty) = \alpha$ and $\xi_{x_0, \beta}(\infty) = \beta$. Then the metric $d_{\partial X}(\alpha, \beta)$ of α and β on ∂X (with respect to the basepoint x_0) is defined by

$$d_{\partial X}(\alpha, \beta) = \sum_{i=1}^{\infty} \min\{d(\xi_{x_0, \alpha}(i), \xi_{x_0, \beta}(i)), \frac{1}{2^i}\}.$$

The metric $d_{\partial X}$ depends on the basepoint x_0 and the topology of ∂X does not depend on x_0 .

The boundary ∂X is said to be *minimal*, if any orbit $G\alpha$ is dense in ∂X . Also the boundary ∂X is called a *scrambled set*, if for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$,

$$\begin{aligned} \limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} &> 0 \text{ and} \\ \liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} &= 0. \end{aligned}$$

We note that “minimality” and “scrambled sets” in this paper is a natural extension of the original definitions in the chaotic theory and original minimality and scrambled sets are important concept on dynamical systems and relate to the chaotic theory in the sense of Li and Yorke (cf. [19], [21] and [22]). Boundaries of CAT(0) groups (and Coxeter groups) are interesting object. In general, they are so complex and it is so difficult to see these topology and the actual actions of CAT(0) groups on their boundaries. The purpose of this paper is to get something information of these actions and boundaries by using a method of the chaotic theory. We can find recent research on minimality of boundaries of CAT(0) groups and Coxeter groups in [12], [14], [16] and [18]. In this paper, we investigate CAT(0) groups and Coxeter groups whose boundaries are scrambled sets.

After some preliminaries on CAT(0) spaces and their boundaries in Section 2, we first show the following theorem in Section 3.

Theorem 1. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. Then*

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

This theorem implies that the boundary ∂X is a scrambled set if and only if

$$\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

In Section 3, we prove a technical theorem which gives a sufficient condition of CAT(0) groups whose boundaries are scrambled sets and which plays a key role in the proof of the main results in this paper.

In Section 4, we study boundaries of hyperbolic CAT(0) groups and we show the following theorems.

Theorem 2. *The boundary of every non-elementary hyperbolic CAT(0) group is a scrambled set.*

An action of a group G on a metric space Y by homeomorphisms is said to be *expansive*, if there exists a constant $c > 0$ such that for each pair $y, y' \in Y$ with $y \neq y'$, there is $g \in G$ such that $d(gy, gy') > c$.

Theorem 3. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. The action of G on ∂X is expansive if and only if the space X is hyperbolic.*

In Section 5, we investigate when are boundaries of CAT(0) groups scrambled sets, and we give a sufficient condition of CAT(0) groups whose boundaries are scrambled sets and also give a sufficient condition of CAT(0) groups whose boundaries are *not* scrambled sets.

In Sections 6, 7 and 8, we study the boundary of a Coxeter system. Definitions and basic properties of Coxeter systems and Coxeter groups are found in [1] and [20]. Every Coxeter system (W, S) determines a *Davis complex* $\Sigma(W, S)$ which is a CAT(0) space ([6], [7], [8], [24]) and the natural action of the Coxeter group W on the Davis complex $\Sigma(W, S)$ is proper, cocompact and by isometries (hence Coxeter groups are CAT(0) groups). The boundary $\partial\Sigma(W, S)$ is called the *boundary* of the Coxeter system (W, S) .

We show a technical theorem which gives a sufficient condition of a Coxeter system whose boundary is a scrambled set in Section 7.

Using the technical theorem, we show the following strong theorem for right-angled Coxeter groups and their boundaries in Section 8.

Theorem 4. *If (W, S) is an irreducible right-angled Coxeter system and $|\partial\Sigma(W, S)| > 2$, then the boundary $\partial\Sigma(W, S)$ is a scrambled set.*

From Theorem 4 and [18, Theorem 5.1], we obtain the following corollary which gives equivalent conditions of a right-angled Coxeter system whose boundary is a scrambled set.

Corollary 5. *Let (W, S) be a right-angled Coxeter system with $|\partial\Sigma(W, S)| > 2$. Then the following statements are equivalent:*

- (1) $\partial\Sigma(W, S)$ is a scrambled set.
- (2) $\partial\Sigma(W, S)$ is minimal.

(3) $(W_{\tilde{S}}, \tilde{S})$ is irreducible.

Here $W_{\tilde{S}}$ is the minimum parabolic subgroup of finite index in (W, S) , that is, for the irreducible decomposition $W = W_{S_1} \times \cdots \times W_{S_n}$, we define $\tilde{S} = \bigcup \{S_i \mid W_{S_i} \text{ is infinite}\}$ ([11]) and $W_{\tilde{S}}$ is the subgroup of W generated by \tilde{S} .

By Corollary 5, we can determine the class of right-angled Coxeter systems whose boundaries are scrambled sets.

2. CAT(0) SPACES AND THEIR BOUNDARIES

We say that a metric space (X, d) is a *geodesic space* if for each $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \rightarrow X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). Also a metric space X is said to be *proper* if every closed metric ball is compact.

Let X be a geodesic space and let T be a geodesic triangle in X . A *comparison triangle* for T is a geodesic triangle \bar{T} in the Euclidean plane \mathbb{R}^2 with same edge lengths as T . Choose two points x and y in T . Let \bar{x} and \bar{y} denote the corresponding points in \bar{T} . Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the *CAT(0)-inequality*, where $d_{\mathbb{R}^2}$ is the usual metric on \mathbb{R}^2 . A geodesic space X is called a *CAT(0) space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T .

Let X be a proper CAT(0) space and $x_0 \in X$. The *boundary of X with respect to x_0* , denoted by $\partial_{x_0} X$, is defined as the set of all geodesic rays issuing from x_0 . Then we define a topology on $X \cup \partial_{x_0} X$ by the following conditions:

- (1) X is an open subspace of $X \cup \partial_{x_0} X$.
- (2) For $\alpha \in \partial_{x_0} X$ and $r, \epsilon > 0$, let

$$U_{x_0}(\alpha; r, \epsilon) = \{x \in X \cup \partial_{x_0} X \mid x \notin B(x_0, r), d(\alpha(r), \xi_x(r)) < \epsilon\},$$

where $\xi_x : [0, d(x_0, x)] \rightarrow X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0} X$). Then for each $\epsilon_0 > 0$, the set

$$\{U_{x_0}(\alpha; r, \epsilon_0) \mid r > 0\}$$

is a neighborhood basis for α in $X \cup \partial_{x_0} X$.

This is called the *cone topology* on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of X ([2], [9]).

Let X be a geodesic space. Two geodesic rays $\xi, \zeta : [0, \infty) \rightarrow X$ are said to be *asymptotic* if there exists a constant N such that $d(\xi(t), \zeta(t)) \leq N$ for any $t \geq 0$. It is known that for each geodesic

ray ξ in X and each point $x \in X$, there exists a unique geodesic ray ξ' issuing from x such that ξ and ξ' are asymptotic.

Let x_0 and x_1 be two points of a proper CAT(0) space X . Then there exists a unique bijection $\Phi : \partial_{x_0}X \rightarrow \partial_{x_1}X$ such that ξ and $\Phi(\xi)$ are asymptotic for any $\xi \in \partial_{x_0}X$. It is known that $\Phi : \partial_{x_0}X \rightarrow \partial_{x_1}X$ is a homeomorphism ([2], [9]).

Let X be a proper CAT(0) space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in X . The *boundary of X* , denoted by ∂X , is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. For each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0}X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0}X$ for each $x_0 \in X$.

We can define a metric on the boundary ∂X as follows: We first fix a basepoint $x_0 \in X$. Let $\alpha, \beta \in \partial X$ and let $\xi_{x_0, \alpha}$ and $\xi_{x_0, \beta}$ be the geodesic rays in X with $\xi_{x_0, \alpha}(0) = \xi_{x_0, \beta}(0) = x_0$, $\xi_{x_0, \alpha}(\infty) = \alpha$ and $\xi_{x_0, \beta}(\infty) = \beta$. Then the metric $d_{\partial X}(\alpha, \beta)$ of α and β on ∂X is defined by

$$d_{\partial X}(\alpha, \beta) = \sum_{i=1}^{\infty} \min\{d(\xi_{x_0, \alpha}(i), \xi_{x_0, \beta}(i)), \frac{1}{2^i}\}.$$

We note that the metric $d_{\partial X}$ depends on the basepoint x_0 .

In this paper, we suppose that every CAT(0) space X has a fixed basepoint x_0 and the metric $d_{\partial X}$ on the boundary ∂X is defined by the basepoint x_0 .

Let X be a proper CAT(0) space and let G be a group which acts on X by isometries. For each element $g \in G$ and each geodesic ray $\xi : [0, \infty) \rightarrow X$, a map $g\xi : [0, \infty) \rightarrow X$ defined by $(g\xi)(t) := g(\xi(t))$ is also a geodesic ray. If geodesic rays ξ and ξ' are asymptotic, then $g\xi$ and $g\xi'$ are also asymptotic. Thus g induces a homeomorphism of ∂X , and the group G acts by homeomorphisms on the boundary ∂X .

A *geometric* action on a CAT(0) space is an action by isometries which is proper ([2, p.131]) and cocompact. We note that every CAT(0) space on which a group acts geometrically is a proper space ([2, p.132]). A group which acts geometrically on some CAT(0) space is called a *CAT(0) group*.

Details of CAT(0) spaces and their boundaries are found in [2] and [9].

Here we introduce some properties of CAT(0) spaces and their boundaries used later.

Lemma 2.1 ([2], [9]). *Let X be a proper CAT(0) space.*

- (1) For each three points $x_0, x_1, x_2 \in X$ and each $t \in [0, 1]$,

$$d(\xi_1(td(x_0, x_1)), \xi_2(td(x_0, x_2))) \leq td(x_1, x_2),$$
where $\xi_i : [0, d_i] \rightarrow X$ is the geodesic segment from x_0 to x_i for each $i = 1, 2$.
- (2) If geodesic rays $\xi, \xi' : [0, \infty) \rightarrow X$ are asymptotic, then

$$d(\xi(t), \xi'(t)) \leq d(\xi(0), \xi'(0))$$
for any $t \geq 0$.

From Lemma 2.1 (1), we obtain the following.

Lemma 2.2 ([2], [9]). *Let X be a $CAT(0)$ space and let $\xi : [0, \infty) \rightarrow X$ and $\xi' : [0, \infty) \rightarrow X$ be two geodesic rays with $\xi(0) = \xi'(0)$. Then for $0 < t \leq t'$,*

$$d(\xi(t), \xi'(t)) \leq \frac{t}{t'} d(\xi(t'), \xi'(t')).$$

We obtain the following lemma from the proof of [13, Lemma 4.2].

Lemma 2.3 ([13, Lemma 4.2]). *Let X be a $CAT(0)$ space and let $\xi : [0, \infty) \rightarrow X$ and $\xi' : [0, \infty) \rightarrow X$ be two geodesic rays with $\xi(0) = \xi'(0)$. For $r > \epsilon > 0$, if $d(\xi(r), \text{Im } \xi') \leq \epsilon$ then $d(\xi(r - \epsilon), \xi'(r - \epsilon)) \leq \epsilon$.*

We define the *angle* of two geodesic paths in a $CAT(0)$ space.

Definition 2.4 ([2, p.9 and p.184]). Let X be a $CAT(0)$ space and let $\xi : [0, a] \rightarrow X$ and $\xi' : [0, a'] \rightarrow X$ be two geodesic paths with $\xi(0) = \xi'(0)$. For $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\overline{\Delta}(\xi(0), \xi(t), \xi'(t'))$, and the comparison angle $\overline{\angle}_{\xi(0)}(\xi(t), \xi'(t'))$. The *Alexandrov angle* between the geodesic paths ξ and ξ' is the number $\angle(\xi, \xi') \in [0, \pi]$ defined by

$$\angle(\xi, \xi') = \limsup_{t, t' \rightarrow 0} \overline{\angle}_{\xi(0)}(\xi(t), \xi'(t')).$$

Lemma 2.5 ([2, p.184, Proposition II.3.1]). *Let X be a $CAT(0)$ space and let $\xi : [0, a] \rightarrow X$ and $\xi' : [0, a'] \rightarrow X$ be two geodesic paths with $\xi(0) = \xi'(0)$. Then*

$$\angle(\xi, \xi') = \lim_{t \rightarrow 0} 2 \arcsin \frac{1}{2t} d(\xi(t), \xi'(t)).$$

We define the *angle* of two points in the boundary of a proper $CAT(0)$ space.

Definition 2.6 ([2, p.280]). Let X be a proper $CAT(0)$ space, let $x \in X$ and let $\alpha, \beta \in \partial X$. The angle $\angle_x(\alpha, \beta)$ at x between α and β is defined as

$$\angle_x(\alpha, \beta) = \angle_x(\xi_{x,\alpha}, \xi_{x,\beta}),$$

where $\xi_{x,\alpha}$ and $\xi_{x,\beta}$ are the geodesic rays with $\xi_{x,\alpha}(0) = \xi_{x,\beta}(0) = x$, $\xi_{x,\alpha}(\infty) = \alpha$ and $\xi_{x,\beta}(\infty) = \beta$.

Also the angle $\angle(\alpha, \beta)$ between α and β is defined as

$$\angle(\alpha, \beta) = \sup_{x \in X} \angle_x(\alpha, \beta).$$

Lemma 2.7 ([2, p.281, Proposition II.9.8]). *Let X be a proper $CAT(0)$ space, let $x_0 \in X$, let $\alpha, \beta \in \partial X$ and let $\xi_{x_0,\alpha}$ and $\xi_{x_0,\beta}$ be the geodesic rays with $\xi_{x_0,\alpha}(0) = \xi_{x_0,\beta}(0) = x_0$, $\xi_{x_0,\alpha}(\infty) = \alpha$ and $\xi_{x_0,\beta}(\infty) = \beta$.*

(1) *The function $t \mapsto \angle_{\xi_{x_0,\alpha}(t)}(\alpha, \beta)$ is non-decreasing and*

$$\angle(\alpha, \beta) = \lim_{t \rightarrow \infty} \angle_{\xi_{x_0,\alpha}(t)}(\alpha, \beta).$$

$$(2) \quad 2 \sin(\angle(\alpha, \beta)/2) = \lim_{t \rightarrow \infty} \frac{1}{t} d(\xi_{x_0,\alpha}(t), \xi_{x_0,\beta}(t)).$$

$$(3) \quad 2 \sin(\angle_{x_0}(\alpha, \beta)/2) = \lim_{t \rightarrow 0} \frac{1}{t} d(\xi_{x_0,\alpha}(t), \xi_{x_0,\beta}(t)).$$

Here we obtain (3) in the above lemma from Lemma 2.5.

Using Lemmas 2.2 and 2.7, we show a lemma.

Lemma 2.8. *Let X be a proper $CAT(0)$ space, let $x_0 \in X$, let $\alpha, \beta \in \partial X$ and let $\xi_{x_0,\alpha}$ and $\xi_{x_0,\beta}$ be the geodesic rays with $\xi_{x_0,\alpha}(0) = \xi_{x_0,\beta}(0) = x_0$, $\xi_{x_0,\alpha}(\infty) = \alpha$ and $\xi_{x_0,\beta}(\infty) = \beta$. Then*

$$2 \sin(\angle_{x_0}(\alpha, \beta)/2) \leq d(\xi_{x_0,\alpha}(1), \xi_{x_0,\beta}(1)).$$

Proof. By Lemma 2.7 (3),

$$2 \sin(\angle_{x_0}(\alpha, \beta)/2) = \lim_{t \rightarrow 0} \frac{1}{t} d(\xi_{x_0,\alpha}(t), \xi_{x_0,\beta}(t)).$$

Here by Lemma 2.2, for any $0 < t \leq t'$

$$\frac{1}{t} d(\xi_{x_0,\alpha}(t), \xi_{x_0,\beta}(t)) \leq \frac{1}{t'} d(\xi_{x_0,\alpha}(t'), \xi_{x_0,\beta}(t')).$$

Hence

$$\begin{aligned} 2 \sin(\angle_{x_0}(\alpha, \beta)/2) &= \lim_{t \rightarrow 0} \frac{1}{t} d(\xi_{x_0,\alpha}(t), \xi_{x_0,\beta}(t)) \\ &\leq d(\xi_{x_0,\alpha}(1), \xi_{x_0,\beta}(1)). \end{aligned}$$

□

3. A KEY THEOREM ON $CAT(0)$ GROUPS WHOSE BOUNDARIES ARE SCRAMBLED SETS

In this section, we show a key theorem which gives a sufficient condition of $CAT(0)$ groups whose boundaries are scrambled sets.

We first prove the following theorem.

Theorem 3.1. *Suppose that a group G acts geometrically on a $CAT(0)$ space X and $|\partial X| > 2$. Then*

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

Proof. Let $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. Then $\angle(\alpha, \beta) > 0$, since $\alpha \neq \beta$. Let $\xi_{x_0, \alpha}$ be the geodesic ray with $\xi_{x_0, \alpha}(0) = x_0$ and $\xi_{x_0, \alpha}(\infty) = \alpha$ and let $x_i = \xi_{x_0, \alpha}(i)$ for each $i \in \mathbb{N}$. Then by Lemma 2.7 (1), the function $i \mapsto \angle_{x_i}(\alpha, \beta)$ is non-decreasing and the sequence $\{\angle_{x_i}(\alpha, \beta)\}_i$ converges to $\angle(\alpha, \beta) > 0$ as $i \rightarrow \infty$. Hence there exists a number $i_0 \in \mathbb{N}$ such that for any $i \geq i_0$,

$$\angle(\alpha, \beta)/2 \leq \angle_{x_i}(\alpha, \beta) \leq \angle(\alpha, \beta).$$

Let $\xi_{x_i, \alpha}$ and $\xi_{x_i, \beta}$ be the geodesic rays with $\xi_{x_i, \alpha}(0) = \xi_{x_i, \beta}(0) = x_i$, $\xi_{x_i, \alpha}(\infty) = \alpha$ and $\xi_{x_i, \beta}(\infty) = \beta$. By Lemma 2.8, we have that

$$d(\xi_{x_i, \alpha}(1), \xi_{x_i, \beta}(1)) \geq 2 \sin(\angle_{x_i}(\alpha, \beta)/2) \geq 2 \sin(\angle(\alpha, \beta)/4),$$

because $\angle_{x_i}(\alpha, \beta) \geq \angle(\alpha, \beta)/2$. Let $r = 2 \sin(\angle(\alpha, \beta)/4)$. Then

$$d(\xi_{x_i, \alpha}(1), \xi_{x_i, \beta}(1)) \geq r \text{ and}$$

$$d(\xi_{x_i, \alpha}(t), \xi_{x_i, \beta}(t)) \geq rt$$

for any $t \geq 1$ by Lemma 2.2. Since the action of G on X is cocompact and X is proper, $GB(x_0, N) = X$ for some $N > 0$. For each $i \in \mathbb{N}$, there exists $g_i \in G$ such that $d(x_i, g_i x_0) \leq N$. Let $\xi_{g_i x_0, \alpha}$ and $\xi_{g_i x_0, \beta}$ be the geodesic rays with $\xi_{g_i x_0, \alpha}(0) = \xi_{g_i x_0, \beta}(0) = g_i x_0$, $\xi_{g_i x_0, \alpha}(\infty) = \alpha$ and $\xi_{g_i x_0, \beta}(\infty) = \beta$. By Lemma 2.1 (2), $d(\xi_{x_i, \alpha}(t), \xi_{g_i x_0, \alpha}(t)) \leq N$ and $d(\xi_{x_i, \beta}(t), \xi_{g_i x_0, \beta}(t)) \leq N$ for any $t \geq 0$. Hence

$$d(\xi_{g_i x_0, \alpha}(t), \xi_{g_i x_0, \beta}(t)) \geq d(\xi_{x_i, \alpha}(t), \xi_{x_i, \beta}(t)) - 2N \geq rt - 2N$$

for each $t \geq 1$. Let $t_0 = \lceil \frac{2N+1}{r} \rceil + 1$. (We note that the number t_0 depends on just α and β .) Then $rt_0 - 2N \geq 1$ and

$$d(\xi_{g_i x_0, \alpha}(t_0), \xi_{g_i x_0, \beta}(t_0)) \geq rt_0 - 2N \geq 1.$$

Here $g_i \xi_{x_0, g_i^{-1} \alpha} = \xi_{g_i x_0, \alpha}$ and $g_i \xi_{x_0, g_i^{-1} \beta} = \xi_{g_i x_0, \beta}$, since g_i is an isometry of X . Hence for each $i \geq i_0$,

$$\begin{aligned} d_{\partial X}(g_i^{-1} \alpha, g_i^{-1} \beta) &= \sum_{j=1}^{\infty} \min\{d(\xi_{x_0, g_i^{-1} \alpha}(j), \xi_{x_0, g_i^{-1} \beta}(j)), \frac{1}{2^j}\} \\ &= \sum_{j=1}^{\infty} \min\{d(g_i^{-1} \xi_{g_i x_0, \alpha}(j), g_i^{-1} \xi_{g_i x_0, \beta}(j)), \frac{1}{2^j}\} \\ &= \sum_{j=1}^{\infty} \min\{d(\xi_{g_i x_0, \alpha}(j), \xi_{g_i x_0, \beta}(j)), \frac{1}{2^j}\} \\ &\geq \frac{1}{2^{t_0}}, \end{aligned}$$

because $d(\xi_{g_i x_0, \alpha}(t_0), \xi_{g_i x_0, \beta}(t_0)) \geq 1$. Here $\frac{1}{2^{t_0}}$ is a constant which depends on just α and β . Thus we obtain that

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} \geq \frac{1}{2^{t_0}} > 0.$$

□

Now we show a theorem which gives a sufficient condition of CAT(0) groups whose boundaries are scrambled sets. This theorem plays a key role in the proof of the main results in this paper.

Theorem 3.2. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. Assume that there exists a constant $M > 0$ such that for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$, there exist a sequence $\{g_i\} \subset G$ and a point $y_0 \in X$ such that $\{d(y_0, g_i x_0)\}_i \rightarrow \infty$ as $i \rightarrow \infty$ and for any $i \in \mathbb{N}$,*

$$\begin{aligned} \text{Im } \xi_{g_i x_0, \alpha} \cap B(y_0, M) &\neq \emptyset \text{ and} \\ \text{Im } \xi_{g_i x_0, \beta} \cap B(y_0, M) &\neq \emptyset, \end{aligned}$$

where $\xi_{g_i x_0, \alpha}$ and $\xi_{g_i x_0, \beta}$ are the geodesic rays with $\xi_{g_i x_0, \alpha}(0) = \xi_{g_i x_0, \beta}(0) = g_i x_0$, $\xi_{g_i x_0, \alpha}(\infty) = \alpha$ and $\xi_{g_i x_0, \beta}(\infty) = \beta$. Then the boundary ∂X is a scrambled set.

Proof. By Theorem 3.1,

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. Hence it is sufficient to show that

$$\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

Let $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. By the assumption, there exist a sequence $\{g_i\} \subset G$ and a point $y_0 \in X$ such that $\{d(y_0, g_i x_0)\}_i \rightarrow \infty$ as $i \rightarrow \infty$ and

$$\begin{aligned} \text{Im } \xi_{g_i x_0, \alpha} \cap B(y_0, M) &\neq \emptyset \text{ and} \\ \text{Im } \xi_{g_i x_0, \beta} \cap B(y_0, M) &\neq \emptyset \end{aligned}$$

for any $i \in \mathbb{N}$. Here we show that

$$\{d_{\partial X}(g_i^{-1}\alpha, g_i^{-1}\beta)\}_i \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Let $\epsilon > 0$ be a small number and let $j_0 \in \mathbb{N}$ such that

$$\frac{1}{2^{j_0-1}} < \epsilon.$$

Since $\{d(y_0, g_i x_0)\}_i \rightarrow \infty$ as $i \rightarrow \infty$, there exists $i_0 \in \mathbb{N}$ such that for any $i \geq i_0$,

$$d(y_0, g_i x_0) > \frac{2Mj_0(j_0+1)}{\epsilon} + 3M.$$

Let $i \geq i_0$ and let $R_i = d(y_0, g_i x_0)$. We suppose that $\xi_{x_0, g_i^{-1}\alpha}$ and $\xi_{x_0, g_i^{-1}\beta}$ are the geodesic rays with $\xi_{x_0, g_i^{-1}\alpha}(0) = \xi_{x_0, g_i^{-1}\beta}(0) = x_0$, $\xi_{x_0, g_i^{-1}\alpha}(\infty) = g_i^{-1}\alpha$ and $\xi_{x_0, g_i^{-1}\beta}(\infty) = g_i^{-1}\beta$. Then $g_i \xi_{x_0, g_i^{-1}\alpha} = \xi_{g_i x_0, \alpha}$ and $g_i \xi_{x_0, g_i^{-1}\beta} = \xi_{g_i x_0, \beta}$, since g_i is an isometry of X . Here

$$\begin{aligned} d_{\partial X}(g_i^{-1}\alpha, g_i^{-1}\beta) &= \sum_{j=1}^{\infty} \min\{d(\xi_{x_0, g_i^{-1}\alpha}(j), \xi_{x_0, g_i^{-1}\beta}(j)), \frac{1}{2^j}\} \\ &\leq \sum_{j=1}^{j_0} d(\xi_{x_0, g_i^{-1}\alpha}(j), \xi_{x_0, g_i^{-1}\beta}(j)) + \sum_{j=j_0+1}^{\infty} \frac{1}{2^j} \\ &= \sum_{j=1}^{j_0} d(\xi_{g_i x_0, \alpha}(j), \xi_{g_i x_0, \beta}(j)) + \frac{1}{2^{j_0}}. \end{aligned}$$

Now we show that

$$d(\xi_{g_i x_0, \alpha}(j), \xi_{g_i x_0, \beta}(j)) < \frac{2Mj}{R_i - 3M}$$

for any $j \in \mathbb{N}$. By the assumption,

$$\begin{aligned} \text{Im } \xi_{g_i x_0, \alpha} \cap B(y_0, M) &\neq \emptyset \text{ and} \\ \text{Im } \xi_{g_i x_0, \beta} \cap B(y_0, M) &\neq \emptyset. \end{aligned}$$

Then $\xi_{g_i x_0, \alpha}(t_i) \in B(y_0, M)$ for some $t_i \geq 0$. Hence $d(y_0, \xi_{g_i x_0, \alpha}(t_i)) \leq M$. Here

$$\begin{aligned} t_i &= d(g_i x_0, \xi_{g_i x_0, \alpha}(t_i)) \\ &\geq d(g_i x_0, y_0) - d(y_0, \xi_{g_i x_0, \alpha}(t_i)) \\ &\geq d(g_i x_0, y_0) - M \\ &= R_i - M \end{aligned}$$

and

$$\begin{aligned} d(\xi_{g_i x_0, \alpha}(t_i), \text{Im } \xi_{g_i x_0, \beta}) &\leq d(\xi_{g_i x_0, \alpha}(t_i), y_0) + d(y_0, \text{Im } \xi_{g_i x_0, \beta}) \\ &\leq 2M. \end{aligned}$$

By Lemma 2.3,

$$d(\xi_{g_i x_0, \alpha}(t_i - 2M), \xi_{g_i x_0, \beta}(t_i - 2M)) \leq 2M.$$

Since $t_i \geq R_i - M$, we have that

$$d(\xi_{g_i x_0, \alpha}(R_i - 3M), \xi_{g_i x_0, \beta}(R_i - 3M)) \leq 2M.$$

Thus by Lemma 2.2,

$$\begin{aligned} d(\xi_{g_i x_0, \alpha}(j), \xi_{g_i x_0, \beta}(j)) &\leq \frac{j}{R_i - 3M} d(\xi_{g_i x_0, \alpha}(R_i - 3M), \xi_{g_i x_0, \beta}(R_i - 3M)) \\ &\leq \frac{2Mj}{R_i - 3M}. \end{aligned}$$

Hence

$$\begin{aligned} d_{\partial X}(g_i^{-1} \alpha, g_i^{-1} \beta) &\leq \sum_{j=1}^{j_0} d(\xi_{g_i x_0, \alpha}(j), \xi_{g_i x_0, \beta}(j)) + \frac{1}{2^{j_0}} \\ &\leq \sum_{j=1}^{j_0} \frac{2Mj}{R_i - 3M} + \frac{1}{2^{j_0}} \\ &= \frac{2M}{R_i - 3M} \sum_{j=1}^{j_0} j + \frac{1}{2^{j_0}} \\ &= \frac{Mj_0(j_0 + 1)}{R_i - 3M} + \frac{1}{2^{j_0}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

because

$$R_i = d(y_0, g_i x_0) > \frac{2Mj_0(j_0 + 1)}{\epsilon} + 3M \text{ and } \frac{1}{2^{j_0-1}} < \epsilon.$$

Here $\epsilon > 0$ is an arbitrary small number. Thus

$$\{d_{\partial X}(g_i^{-1}\alpha, g_i^{-1}\beta)\}_i \rightarrow 0 \text{ as } i \rightarrow \infty.$$

This implies that

$$\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0.$$

Hence the boundary ∂X is a scrambled set. \square

4. HYPERBOLIC CASE

In this section, we study boundaries of hyperbolic CAT(0) groups.

We first introduce a definition of hyperbolic spaces.

A geodesic space X is called a *hyperbolic space*, if there exists a number $\delta > 0$ such that every geodesic triangle in X is “ δ -thin”.

Here “ δ -thin” is defined as follows: Let $x, y, z \in X$ and let $\triangle := \triangle xyz$ be a geodesic triangle in X . There exist unique non-negative numbers a, b, c such that

$$d(x, y) = a + b, \quad d(y, z) = b + c, \quad d(z, x) = c + a.$$

Then we can consider the metric tree T_\triangle that has three vertexes of valence one, one vertex of valence three, and edges of length a, b and c . Let o be the vertex of valence three in T_\triangle and let v_x, v_y, v_z be the vertexes of T_\triangle such that $d(o, v_x) = a, d(o, v_y) = b$ and $d(o, v_z) = c$. Then the map $\{x, y, z\} \rightarrow \{v_x, v_y, v_z\}$ extends uniquely to a map $f : \triangle \rightarrow T_\triangle$ whose restriction to each side of \triangle is an isometry. For some $\delta \geq 0$, the geodesic triangle \triangle is said to be δ -thin, if $d(p, q) \leq \delta$ for each points $p, q \in \triangle$ with $f(p) = f(q)$.

It is known that a geodesic space X is hyperbolic if and only if there exists a number $\delta > 0$ such that every geodesic triangle in X is “ δ -slim”. Here a geodesic triangle is said to be δ -slim, if each of its sides is contained in the δ -neighbourhood of the union of the other two sides.

For a proper hyperbolic space X , we can define the *boundary* ∂X of X , and if the space X is hyperbolic and CAT(0), then these “boundaries” coincide.

Details and basic properties of hyperbolic spaces and their boundaries are found in [2], [4], [9] and [10].

A group G is called a *hyperbolic group* if the group G acts geometrically on some hyperbolic space X . A hyperbolic group G determines the

boundary ∂X of a hyperbolic space X on which G acts geometrically. The boundary ∂X is called the *boundary of G* and denoted by ∂G .

It is known when is a CAT(0) space hyperbolic.

Theorem 4.1 ([2, p.400, Theorem III.H.1.5]). *A proper cocompact CAT(0) space X is hyperbolic if and only if it does not contain a subspace isometric to the flat plane \mathbb{R}^2 .*

We show that the boundary of a non-elementary hyperbolic CAT(0) group is always a scrambled set.

Theorem 4.2. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. If X is hyperbolic, then the boundary ∂X is a scrambled set. Moreover there exists a constant $c > 0$ such that*

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > c$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

Proof. Suppose that X is hyperbolic. There exists a constant $\delta > 0$ such that every geodesic triangle in X is δ -thin. Since the action of G on X is cocompact and X is proper, $GB(x_0, N) = X$ for some $N > 0$.

Let $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. Since $|\partial X| > 2$, there exists $\gamma \in \partial X \setminus \{\alpha, \beta\}$. Let $x_i = \xi_{x_0, \gamma}(i)$ for each $i \in \mathbb{N}$, where $\xi_{x_0, \gamma}$ is the geodesic ray with $\xi_{x_0, \gamma}(0) = x_0$ and $\xi_{x_0, \gamma}(\infty) = \gamma$. We consider the two triangles of x_0, x_i, α and x_0, x_i, β . Since X is hyperbolic, there exists an enough large number $R > 0$ such that for any $i \in \mathbb{N}$ with $i > R$,

$$\begin{aligned} d(\xi_{x_0, \gamma}(R), \text{Im } \xi_{x_i, \alpha}) &\leq \delta \text{ and} \\ d(\xi_{x_0, \gamma}(R), \text{Im } \xi_{x_i, \beta}) &\leq \delta. \end{aligned}$$

For each $i \in \mathbb{N}$ with $i > R$, there exists $g_i \in G$ such that $d(x_i, g_i x_0) \leq N$ by the definition of the number $N > 0$. Then $d(\xi_{x_0, \gamma}(R), \text{Im } \xi_{x_i, \alpha}) \leq \delta$ and $d(\xi_{x_i, \alpha}(t), \xi_{g_i x_0, \alpha}(t)) \leq N$ for any $t \geq 0$ by Lemma 2.1 (2). Hence

$$d(\xi_{x_0, \gamma}(R), \text{Im } \xi_{g_i x_0, \alpha}) \leq N + \delta.$$

Also we have that

$$d(\xi_{x_0, \gamma}(R), \text{Im } \xi_{g_i x_0, \beta}) \leq N + \delta.$$

Thus the constant $M = N + \delta$, the point $y_0 = \xi_{x_0, \gamma}(R)$ and the sequence $\{g_i \mid i \in \mathbb{N}, i > R\}$ satisfy the condition of Theorem 3.2. Hence the boundary ∂X is a scrambled set.

Moreover, for each $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$, $\angle(\alpha, \beta) = \pi$ because there exists a geodesic line $\sigma : \mathbb{R} \rightarrow X$ such that $\sigma(\infty) = \alpha$ and $\sigma(-\infty) = \beta$

(cf. [2, p.428, Lemma III.H.3.2] and [4]). Hence by the proof of Theorem 3.1, for the constants $r = 2\sin(\pi/4)$ and $t_0 = \lceil \frac{2N+1}{r} \rceil + 1$, there exists a sequence $\{g_i\} \subset G$ such that

$$d_{\partial X}(g_i^{-1}\alpha, g_i^{-1}\beta) \geq \frac{1}{2^{t_0}} > \frac{1}{2^{t_0+1}}.$$

Here $c := \frac{1}{2^{t_0+1}}$ is a constant which does not depend on α and β . Thus

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > c$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. \square

It is known that the boundary ∂G of a hyperbolic group G is minimal. Hence the boundary ∂G of every non-elementary hyperbolic CAT(0) group G is a scrambled set and minimal.

Also we obtain a theorem.

Theorem 4.3. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. The space X is hyperbolic if and only if there exists a constant $c > 0$ such that*

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > c$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

Proof. By Theorem 4.2, it is sufficient to show that if X is not hyperbolic then there does not exist a constant $c > 0$ such that

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > c$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

Suppose that X is not hyperbolic. Then X contains a subspace Z which is isometric to the flat plane \mathbb{R}^2 by Theorem 4.1. The boundary of Z is the circle, and for each $\theta \in [0, \pi]$ there exist $\alpha, \beta \in \partial Z$ with $\angle(\alpha, \beta) = \theta$.

Let $\epsilon > 0$ be a small number. There exists $t_0 \in \mathbb{N}$ such that $\frac{1}{2^{t_0}} < \frac{\epsilon}{2}$. Then there exist $\alpha, \beta \in \partial Z \subset \partial X$ with $\alpha \neq \beta$ such that

$$\sin(\angle(\alpha, \beta)/2) \leq \frac{\epsilon}{4t_0^2}.$$

For each $y_0 \in X$, Lemmas 2.2 and 2.7 (2) imply that

$$\begin{aligned} \frac{1}{t_0} d(\xi_{y_0, \alpha}(t_0), \xi_{y_0, \beta}(t_0)) &\leq \lim_{t \rightarrow \infty} \frac{1}{t} d(\xi_{y_0, \alpha}(t), \xi_{y_0, \beta}(t)) \\ &= 2 \sin(\angle(\alpha, \beta)/2) \\ &\leq 2\left(\frac{\epsilon}{4t_0^2}\right) = \frac{\epsilon}{2t_0^2}, \end{aligned}$$

where $\xi_{y_0, \alpha}$ and $\xi_{y_0, \beta}$ are the geodesic rays with $\xi_{y_0, \alpha}(0) = \xi_{y_0, \beta}(0) = y_0$, $\xi_{y_0, \alpha}(\infty) = \alpha$ and $\xi_{y_0, \beta}(\infty) = \beta$. Hence

$$d(\xi_{y_0, \alpha}(t_0), \xi_{y_0, \beta}(t_0)) \leq \frac{\epsilon}{2t_0}$$

for any $y_0 \in X$. Thus for each $j \in \{1, 2, \dots, t_0\}$,

$$d(\xi_{y_0, \alpha}(j), \xi_{y_0, \beta}(j)) \leq d(\xi_{y_0, \alpha}(t_0), \xi_{y_0, \beta}(t_0)) \leq \frac{\epsilon}{2t_0}$$

by Lemma 2.2.

By the above argument, for each $g \in G$,

$$\begin{aligned} d_{\partial X}(g\alpha, g\beta) &= \sum_{j=1}^{\infty} \min\{d(\xi_{x_0, g\alpha}(j), \xi_{x_0, g\beta}(j)), \frac{1}{2^j}\} \\ &= \sum_{j=1}^{\infty} \min\{d(\xi_{g^{-1}x_0, \alpha}(j), \xi_{g^{-1}x_0, \beta}(j)), \frac{1}{2^j}\} \\ &\leq \sum_{j=1}^{t_0} d(\xi_{g^{-1}x_0, \alpha}(j), \xi_{g^{-1}x_0, \beta}(j)) + \sum_{j=t_0+1}^{\infty} \frac{1}{2^j} \\ &\leq \sum_{j=1}^{t_0} \frac{\epsilon}{2t_0} + \frac{1}{2^{t_0}} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus for any $\epsilon > 0$ there exists $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$ such that

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} \leq \epsilon.$$

This means that there does not exist a constant $c > 0$ such that

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > c$$

for any $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$. \square

An action of a group G on a metric space Y by homeomorphisms is said to be *expansive*, if there exists a constant $c > 0$ such that for each pair $y, y' \in Y$ with $y \neq y'$, there is $g \in G$ such that $d(gy, gy') > c$.

We obtain a corollary from the proofs of Theorems 4.2 and 4.3.

Corollary 4.4. *Suppose that a group G acts geometrically on a $CAT(0)$ space X and $|\partial X| > 2$. The action of G on ∂X is expansive if and only if the space X is hyperbolic.*

5. $CAT(0)$ GROUPS WHOSE BOUNDARIES ARE SCRAMBLED SETS

In this section, we investigate when are boundaries of $CAT(0)$ groups scrambled sets.

We first give a sufficient condition of $CAT(0)$ groups whose boundaries are scrambled sets.

Suppose that a group G acts geometrically on a $CAT(0)$ space X . For an element $g \in G$, we define Z_g as the centralizer of g and define F_g as the fixed-point set of g in X , that is,

$$Z_g = \{h \in G \mid gh = hg\} \text{ and } F_g = \{x \in X \mid gx = x\}.$$

The following lemmas are known.

Lemma 5.1 ([16, Theorem 2.1] and [26]). *Suppose that a group G acts geometrically on a $CAT(0)$ space X . For $g \in G$, if Z_g is finite then F_g is bounded.*

Lemma 5.2 ([2, Proposition II.6.2(2)]). *Let X be a $CAT(0)$ space and let g and h be isometries of X . Then $gF_h = F_{ghg^{-1}}$.*

Here we show a theorem.

Theorem 5.3. *Suppose that a group G acts geometrically on a $CAT(0)$ space X . If there exists an element $g_0 \in G$ such that*

- (1) Z_{g_0} is finite,
- (2) $X \setminus F_{g_0}$ is not connected, and
- (3) each component of $X \setminus F_{g_0}$ is convex and not g_0 -invariant,

then the boundary ∂X is a scrambled set.

Proof. Suppose that there exists an element $g_0 \in G$ which satisfies the conditions (1), (2) and (3). Using Theorem 3.2, we show that the boundary ∂X is a scrambled set.

Let $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$.

By (1) and Lemma 5.1, F_{g_0} is bounded. Hence for some (and any) $y_0 \in F_{g_0}$,

$$L(\bigcup \{gF_{g_0} \mid g \in G\}) = L(Gy_0) = \partial X.$$

Here for a subset $A \subset X$, the limit set $L(A)$ of A is defined as

$$L(A) = \overline{A} \cap \partial X,$$

where \overline{A} is the closure of A in $X \cup \partial X$.

Now $|\partial X| > 2$ (hence ∂X is uncountable). Then there exists $h \in G$ such that

$$(\text{Im } \xi_{x_0, \alpha} \cup \text{Im } \xi_{x_0, \beta}) \cap hF_{g_0} = \emptyset.$$

Indeed if

$$(\text{Im } \xi_{x_0, \alpha} \cup \text{Im } \xi_{x_0, \beta}) \cap gF_{g_0} \neq \emptyset$$

for any $g \in G$, then

$$\{\alpha, \beta\} = L(\text{Im } \xi_{x_0, \alpha} \cup \text{Im } \xi_{x_0, \beta}) = \partial X,$$

which contradicts to $|\partial X| > 2$.

Let $h \in G$ such that

$$(\text{Im } \xi_{x_0, \alpha} \cup \text{Im } \xi_{x_0, \beta}) \cap hF_{g_0} = \emptyset.$$

By Lemma 5.2, $hF_{g_0} = F_{hg_0h^{-1}}$. We consider the point $hg_0h^{-1}x_0$. By (2) and (3), $X \setminus hF_{g_0}$ is not connected, and two points x_0 and $hg_0h^{-1}x_0$ are in distinct components of $X \setminus hF_{g_0}$. Let A_0 and A_1 be the components of $X \setminus hF_{g_0}$ such that $x_0 \in A_0$ and $hg_0h^{-1}x_0 \in A_1$. Then $\text{Im } \xi_{x_0, \alpha} \cup \text{Im } \xi_{x_0, \beta} \subset A_0$. Since the components A_0 and A_1 are unbounded, we can take a sequence $\{h_i\} \subset G$ such that $\{h_ix_0\}_i \subset A_1$ and $\{d(h_ix_0, hF_{g_0})\}_i \rightarrow \infty$ as $i \rightarrow \infty$. Since $h_ix_0 \in A_1$ and $\alpha, \beta \in \partial A_0$,

$$\text{Im } \xi_{h_ix_0, \alpha} \cap hF_{g_0} \neq \emptyset \text{ and}$$

$$\text{Im } \xi_{h_ix_0, \beta} \cap hF_{g_0} \neq \emptyset$$

for each i . For the diameter $M = \text{diam}(F_{g_0}) = \text{diam}(hF_{g_0})$ and some point $y_0 \in hF_{g_0}$, we have that

$$\text{Im } \xi_{h_ix_0, \alpha} \cap B(y_0, M) \neq \emptyset \text{ and}$$

$$\text{Im } \xi_{h_ix_0, \beta} \cap B(y_0, M) \neq \emptyset.$$

Here $M > 0$ is a constant which does not depend on α and β .

Thus the condition of Theorem 3.2 holds, and the boundary ∂X is a scrambled set. \square

It is known that if the condition of Theorem 5.3 holds then the boundary ∂X is also minimal ([16]).

In this paper, we define a *reflection* of a geodesic space as follows: An isometry r of a geodesic space X is called a *reflection* of X , if

- (1) r^2 is the identity of X ,
- (2) $X \setminus F_r$ has exactly two convex connected components X_r^+ and X_r^- and
- (3) $rX_r^+ = X_r^-$,

where F_r is the fixed-points set of r . We note that “reflections” in this paper need not satisfy the condition (4) $\text{Int } F_r = \emptyset$ in [15].

We obtain a corollary from Theorem 5.3.

Corollary 5.4. *Suppose that a group G acts geometrically on a $\text{CAT}(0)$ space X and $|\partial X| > 2$. If there exists a reflection $r \in G$ of X such that Z_r is finite, then the boundary ∂X is a scrambled set.*

Next, we give a sufficient condition of $\text{CAT}(0)$ groups whose boundaries are *not* scrambled sets.

A subset A of a metric space Y is said to be *quasi-dense*, if there exists a constant $K > 0$ such that for each $y \in Y$ there is $a \in A$ such that $d(y, a) \leq K$, that is,

$$B(A, K) = \{y \in Y \mid d(y, a) \leq K \text{ for some } a \in A\} = Y.$$

Theorem 5.5. *Suppose that a group G acts geometrically on a $\text{CAT}(0)$ space X and $|\partial X| > 2$. If X contains a quasi-dense subset $X_1 \times X_2$ such that X_1 and X_2 are unbounded, then the boundary ∂X is not a scrambled set.*

Proof. Suppose that X contains a quasi-dense subset $X_1 \times X_2$ such that X_1 and X_2 are unbounded. Then there exists a constant $K > 0$ such that

$$B(X_1 \times X_2, K) = X.$$

Since X_1 and X_2 are unbounded, there exist $\alpha \in \partial X_1$ and $\beta \in \partial X_2$. We note that

$$\partial X = \partial(X_1 \times X_2) = \partial X_1 * \partial X_2,$$

where $\partial X_1 * \partial X_2$ is the spherical join. Hence $\angle(\alpha, \beta) = \pi/2$ and $\angle_{z_0}(\alpha, \beta) = \pi/2$ for each $z_0 \in X_1 \times X_2$.

To show that the boundary ∂X is not a scrambled set, we prove that

$$\liminf \{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0,$$

that is, we show that there exists a constant $c > 0$ such that $d_{\partial X}(g\alpha, g\beta) \geq c$ for any $g \in G$.

Let $t_0 = \lceil \frac{2K+1}{\sqrt{2}} \rceil + 1$ and $c = \frac{1}{2^{t_0}}$. Then we show that $d_{\partial X}(g\alpha, g\beta) \geq c$ for any $g \in G$.

Let $g \in G$. Since $B(X_1 \times X_2, K) = X$, there exists $z_0 \in X_1 \times X_2$ such that $d(g^{-1}x_0, z_0) \leq K$. We consider the geodesic rays $\xi_{z_0, \alpha}$, $\xi_{z_0, \beta}$, $\xi_{g^{-1}x_0, \alpha}$ and $\xi_{g^{-1}x_0, \beta}$. By Lemma 2.1 (2),

$$d(\xi_{g^{-1}x_0, \alpha}(t), \xi_{z_0, \alpha}(t)) \leq K \text{ and}$$

$$d(\xi_{g^{-1}x_0, \beta}(t), \xi_{z_0, \beta}(t)) \leq K.$$

Since $\angle_{z_0}(\alpha, \beta) = \pi/2$ and the convex hull of $\text{Im } \xi_{z_0, \alpha} \cup \text{Im } \xi_{z_0, \beta}$ is flat, we have that $d(\xi_{z_0, \alpha}(t), \xi_{z_0, \beta}(t)) = \sqrt{2}t$ for any $t \geq 0$. Hence

$$\begin{aligned} d(\xi_{g^{-1}x_0, \alpha}(t), \xi_{g^{-1}x_0, \beta}(t)) &\geq d(\xi_{z_0, \alpha}(t), \xi_{z_0, \beta}(t)) - 2K \\ &= \sqrt{2}t - 2K. \end{aligned}$$

Here $\sqrt{2}t - 2K \geq 1$ if and only if $t \geq \frac{2K+1}{\sqrt{2}}$. Hence for $t_0 = \lceil \frac{2K+1}{\sqrt{2}} \rceil + 1$, we obtain that

$$d(\xi_{g^{-1}x_0, \alpha}(t_0), \xi_{g^{-1}x_0, \beta}(t_0)) \geq \sqrt{2}t_0 - 2K \geq 1.$$

Then

$$\begin{aligned} d_{\partial X}(g\alpha, g\beta) &= \sum_{j=1}^{\infty} \min\{d(\xi_{x_0, g\alpha}(j), \xi_{x_0, g\beta}(j)), \frac{1}{2^j}\} \\ &= \sum_{j=1}^{\infty} \min\{d(\xi_{g^{-1}x_0, \alpha}(j), \xi_{g^{-1}x_0, \beta}(j)), \frac{1}{2^j}\} \\ &\geq \frac{1}{2^{t_0}} = c. \end{aligned}$$

Here $g \in G$ is arbitrary. Thus

$$\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} \geq c > 0,$$

and the boundary ∂X is not a scrambled set. \square

By a splitting theorem on CAT(0) spaces ([17] and [23], cf. [18]), we obtain that if a CAT(0) group G contains a subgroup $G_1 \times G_2$ of finite index such that G_1 and G_2 are infinite, then a CAT(0) space X on which G acts geometrically contains a quasi-dense subspace which splits as a product $X_1 \times X_2$, where X_1 and X_2 are unbounded. This implies the following corollary.

Corollary 5.6. *Suppose that a group G acts geometrically on a CAT(0) space X and $|\partial X| > 2$. If G contains a subgroup $G_1 \times G_2$ of finite index such that G_1 and G_2 are infinite, then the boundary ∂X is not a scrambled set.*

6. COXETER GROUPS AND DAVIS COMPLEXES

In this section, we introduce definitions and some properties of Coxeter groups and Davis complexes.

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

- (1) $m(s, t) = m(t, s)$ for any $s, t \in S$,
- (2) $m(s, s) = 1$ for any $s \in S$, and
- (3) $m(s, t) \geq 2$ for any $s, t \in S$ with $s \neq t$.

The pair (W, S) is called a *Coxeter system*. If, in addition,

- (4) $m(s, t) = 2$ or ∞ for any $s, t \in S$ with $s \neq t$,

then (W, S) is said to be *right-angled*. We note that for $s, t \in S$, $m(s, t) = 2$ if and only if $st = ts$.

Let (W, S) be a Coxeter system. For $w \in W$, we denote by $\ell(w)$ the word length of w with respect to S . For $w \in W$, a representation $w = s_1 \cdots s_l$ ($s_i \in S$) is said to be *reduced*, if $\ell(w) = l$. The Coxeter group W has the *word metric* d_ℓ defined by $d_\ell(w, w') = \ell(w^{-1}w')$ for each $w, w' \in W$. Also for a subset $T \subset S$, W_T is defined as the subgroup of W generated by T , and called a *parabolic subgroup*. It is known that the pair (W_T, T) is also a Coxeter system ([1]). If T is the empty set, then W_T is the trivial group. A subset $T \subset S$ is called a *spherical subset* of S , if the parabolic subgroup W_T is finite.

Let (W, S) be a Coxeter system. For each $w \in W$, we define a subset $S(w)$ of S as

$$S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}.$$

Also for a subset T of S , we define a subset W^T of W as

$$W^T = \{w \in W \mid S(w) = T\}.$$

The following lemma is known.

Lemma 6.1 ([1], [6], [8]). *Let (W, S) be a Coxeter system. For each $w \in W$, $S(w)$ is a spherical subset of S , i.e., $W_{S(w)}$ is finite.*

Every Coxeter system (W, S) determines a *Davis complex* $\Sigma(W, S)$ which is a CAT(0) geodesic space ([6], [7], [8], [24]). Here the vertex set of $\Sigma(W, S)$ is W and the 1-skeleton of $\Sigma(W, S)$ is the Cayley graph of W with respect to S . Also $\Sigma(W, S)$ is contractible. The natural action of W on $\Sigma(W, S)$ is proper, cocompact and by isometries, i.e., the Coxeter group W acts geometrically on the Davis complex $\Sigma(W, S)$ and W is a CAT(0) group. If W is infinite, then $\Sigma(W, S)$ is noncompact and we can consider the boundary $\partial\Sigma(W, S)$ of the CAT(0) space $\Sigma(W, S)$. This boundary $\partial\Sigma(W, S)$ is called the *boundary of (W, S)* .

Let (W, S) be a Coxeter system. The set $R_S = \{wsw^{-1} \mid w \in W, s \in S\}$ is called the set of *reflections* of (W, S) . In fact, each $r \in R_S$ is a “reflection” of the Davis complex $\Sigma(W, S)$. We define $K(W, S)$ as the closure of the component C of $\Sigma(W, S) \setminus \bigcup_{r \in R_S} F_r$ with $1 \in C$, where

F_r is the fixed-point set of r in $\Sigma(W, S)$. It is known that the subset $K(W, S)$ is compact and a fundamental domain of the action of W on $\Sigma(W, S)$, that is, $WK(W, S) = \Sigma(W, S)$.

The following lemmas are known. These lemmas give a relation between geodesic paths in Davis complexes and reduced words in Coxeter systems.

Lemma 6.2 ([11, Lemma 4.2]). *Let (W, S) be a Coxeter system and let N be the diameter of $K(W, S)$ in $\Sigma(W, S)$. Then for any $w \in W$ with $w \neq 1$, there exists a reduced representation $w = s_1 \cdots s_l$ such that*

$$d(s_1 \cdots s_i, [1, w]) \leq N$$

for any $i \in \{1, \dots, l\}$, where $[1, w]$ is the geodesic from 1 to w in $\Sigma(W, S)$.

Lemma 6.3 ([14, Lemma 2.6]). *Let (W, S) be a Coxeter system and let N be the diameter of $K(W, S)$ in $\Sigma(W, S)$. Then for any $\alpha \in \partial\Sigma(W, S)$ there exists a sequence $\{s_i\} \subset S$ such that $s_1 \cdots s_i$ is reduced and*

$$d(s_1 \cdots s_i, \text{Im } \xi_{1, \alpha}) \leq N$$

for any $i \in \mathbb{N}$, where $\xi_{1, \alpha}$ is the geodesic ray in $\Sigma(W, S)$ with $\xi_{1, \alpha}(0) = 1$ and $\xi_{1, \alpha}(\infty) = \alpha$.

Also there is the following lemma.

Lemma 6.4 ([14, Lemma 3.3]). *Let (W, S) be a Coxeter system, let N be the diameter of $K(W, S)$ in $\Sigma(W, S)$ and let $x, y \in W$. If $o(st) = \infty$ for each $s \in S(x)$ and $t \in S(y^{-1})$, then $d(x, [1, xy]) \leq N$, where $o(st)$ is the order of st in W .*

7. COXETER SYSTEMS WHOSE BOUNDARIES ARE SCRAMBLED SETS

In this section, we investigate Coxeter systems whose boundaries are scrambled sets. We give sufficient conditions of a Coxeter system whose boundary is a scrambled set.

Theorem 7.1. *Let (W, S) be a Coxeter system with $|\partial\Sigma(W, S)| > 2$. Suppose that there exist $s_0, t_0 \in S$ and a number $K > 0$ such that*

- (1) $o(s_0 t_0) = \infty$ and
- (2) *for each $w, v \in W$, there exists $x \in W$ such that $\ell(x) \leq K$ and $wx, vx \in W^{\{s_0\}}$.*

Then the boundary $\partial\Sigma(W, S)$ is a scrambled set.

Proof. Let $\alpha, \beta \in \partial\Sigma(W, S)$ with $\alpha \neq \beta$. By Lemma 6.3, there exist sequences $\{a_i\}, \{b_i\} \subset S$ such that

$$\begin{aligned} d(a_1 \cdots a_i, \text{Im } \xi_{1,\alpha}) &\leq N \text{ and} \\ d(b_1 \cdots b_i, \text{Im } \xi_{1,\beta}) &\leq N \end{aligned}$$

for each $i \in \mathbb{N}$, where $N = \text{diam}(K(W, S))$. Let $w_i = a_1 \cdots a_i$ and $v_i = b_1 \cdots b_i$ for each i . By (2), for each i , there exists $x_i \in W$ such that $\ell(x_i) \leq K$ and $w_i^{-1}x_i, v_i^{-1}x_i \in W^{\{s_0\}}$. Since $\{x \in W \mid \ell(x) \leq K\}$ is finite, there exist $x \in W$ and a subsequence $\{i_j \mid j \in \mathbb{N}\} \subset \mathbb{N}$ such that $x_{i_j} = x$ for any $j \in \mathbb{N}$. We note that the sequences $\{x^{-1}w_{i_j}\}$ and $\{x^{-1}v_{i_j}\}$ converge to $x^{-1}\alpha$ and $x^{-1}\beta$ respectively, $(x^{-1}w_{i_j})^{-1} = w_{i_j}^{-1}x \in W^{\{s_0\}}$ and $(x^{-1}v_{i_j})^{-1} = v_{i_j}^{-1}x \in W^{\{s_0\}}$. By Lemma 6.4,

$$d((s_0t_0)^k, [1, (s_0t_0)^k x^{-1}w_{i_j}]) \leq N$$

for each $k \in \mathbb{N}$ and $j \in \mathbb{N}$, because $o(s_0t_0) = \infty$ by (1) and $(s_0t_0)^k \in W^{\{t_0\}}$. Hence

$$d((s_0t_0)^k, \text{Im } \xi_{1, (s_0t_0)^k x^{-1}\alpha}) \leq N$$

for each $k \in \mathbb{N}$. Let $g_k = (s_0t_0)^k x^{-1}$ for each $k \in \mathbb{N}$. Since g_k^{-1} is an isometry of $\Sigma(W, S)$, we have that

$$d(g_k^{-1}(s_0t_0)^k, \text{Im } \xi_{g_k^{-1}, g_k^{-1}(s_0t_0)^k x^{-1}\alpha}) \leq N.$$

Hence

$$d(x, \text{Im } \xi_{g_k^{-1}, \alpha}) \leq N$$

for each $k \in \mathbb{N}$. By the same argument, we also obtain that

$$d(x, \text{Im } \xi_{g_k^{-1}, \beta}) \leq N$$

for each $k \in \mathbb{N}$. Here $d(1, x) \leq K$ because $\ell(x) \leq K$. Hence

$$\begin{aligned} d(1, \text{Im } \xi_{g_k^{-1}, \alpha}) &\leq N + K \text{ and} \\ d(1, \text{Im } \xi_{g_k^{-1}, \beta}) &\leq N + K \end{aligned}$$

for each $k \in \mathbb{N}$. We note that $\{d(1, g_k^{-1})\}_k \rightarrow \infty$ as $k \rightarrow \infty$ and the number $M := N + K$ is a constant which does not depend on α and β .

Thus the condition of Theorem 3.2 holds, and the boundary $\partial\Sigma(W, S)$ is a scrambled set \square

There is a similarity between the conditions of Theorem 7.1 and [18, Theorem 3.1] (and [12, Theorem 4.1]). Here [18, Theorem 3.1] and [12, Theorem 4.1] give a sufficient condition of Coxeter systems whose boundaries are minimal.

For a Coxeter system (W, S) and the Davis complex $\Sigma(W, S)$, each $s \in S$ is a “reflection” of $\Sigma(W, S)$ in the sense of the definition in Section 5. Hence we obtain a corollary from Corollary 5.4.

Corollary 7.2. *Let (W, S) be a Coxeter system with $|\partial\Sigma(W, S)| > 2$. If there exists $s \in S$ such that Z_s is finite, then the boundary $\partial\Sigma(W, S)$ is a scrambled set.*

We give an example of a Coxeter group which is not hyperbolic and which satisfies the condition in Corollary 7.2 (hence this example satisfies the conditions in Theorem 5.3 and Corollary 5.4).

Example 7.3. We consider the Coxeter system (W, S) defined by the diagram in Figure 1. Since $\Sigma(W_{\{t_1, t_2, t_3\}}, \{t_1, t_2, t_3\})$ is the flat Euclidean plane, the Coxeter group W is not hyperbolic. Also $Z_s = W_{\{s, t_1, t_2\}}$ is finite. Hence (W, S) satisfies the condition of Corollary 7.2, and the boundary $\partial\Sigma(W, S)$ is a scrambled set.

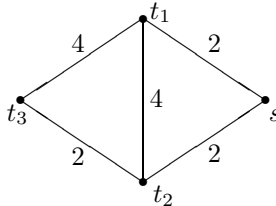


FIGURE 1.

8. RIGHT-ANGLED COXETER GROUPS

The purpose of this section is to prove the following theorem.

Theorem 8.1. *If (W, S) is an irreducible right-angled Coxeter system and $|\partial\Sigma(W, S)| > 2$, then the boundary $\partial\Sigma(W, S)$ is a scrambled set.*

A Coxeter system (W, S) is said to be *irreducible* if, for any nonempty and proper subset T of S , W does not decompose as the direct product of W_T and $W_{S \setminus T}$.

A Coxeter group W is said to be *right-angled*, if (W, S) is a right-angled Coxeter system for some $S \subset W$. It is known that every right-angled Coxeter group determines its Coxeter system up to isomorphisms ([25]). Hence a right-angled Coxeter group W determines the boundary $\partial\Sigma(W, S)$.

The following lemmas are known.

Lemma 8.2 ([1], [20]). *Let (W, S) be a right-angled Coxeter system.*

- (1) W is finite if and only if $st = ts$ for any $s, t \in S$, that is, W is isomorphic to $(\mathbb{Z}_2)^{|S|}$.
- (2) (W, S) is irreducible if and only if for each $a, b \in S$ with $a \neq b$ there exists a sequence $\{a = s_1, s_2, \dots, s_n = b\} \subset S$ such that $o(s_i s_{i+1}) = \infty$ for any $i \in \{1, \dots, n-1\}$.

Lemma 8.3 ([18, Lemma 2.7]). *Let (W, S) be a right-angled Coxeter system, let U be a spherical subset of S , let $s_0 \in S \setminus U$ and let $T = \{t \in U \mid s_0 t = t s_0\}$. Then $W^U s_0 \subset W^{T \cup \{s_0\}}$, that is, $S(ws_0) = T \cup \{s_0\}$ for any $w \in W^U$.*

We first show two technical lemmas.

Lemma 8.4. *Let (W, S) be an irreducible right-angled Coxeter system and let $w \in W$. Suppose that $t_1, t_2, \dots, t_n \in S$ such that*

- (1) $t_1 \notin S(w)$,
- (2) $o(t_i t_{i+1}) = \infty$ for any $i \in \{1, \dots, n-1\}$, and
- (3) $\{t_1, t_2, \dots, t_n\} = S$.

Then $w(t_1 t_2 \dots t_n) \in W^{\{t_n\}}$.

Proof. By Lemma 8.3, we have that

$$S(wt_1) = \{t \in S(w) \mid tt_1 = t_1 t\} \cup \{t_1\} = T_1 \cup \{t_1\},$$

where $T_1 = \{t \in S(w) \mid tt_1 = t_1 t\}$. Also

$$\begin{aligned} S(wt_1 t_2) &= \{t \in S(wt_1) \mid tt_2 = t_2 t\} \cup \{t_2\} \\ &= \{t \in T_1 \cup \{t_1\} \mid tt_2 = t_2 t\} \cup \{t_2\} \\ &= \{t \in T_1 \mid tt_2 = t_2 t\} \cup \{t_2\} \\ &= \{t \in S(w) \mid tt_1 = t_1 t, tt_2 = t_2 t\} \cup \{t_2\}. \end{aligned}$$

By iterating the same argument, we obtain that

$$S(w(t_1 \dots t_n)) = \{t \in S(w) \mid tt_i = t_i t \text{ for any } i \in \{1, \dots, n\}\} \cup \{t_n\}.$$

Here $\{t_1, t_2, \dots, t_n\} = S$ by (3). Since (W, S) is irreducible, there does not exist $t \in S$ such that $tt_i = t_i t$ for any $i \in \{1, \dots, n\}$. Hence $S(w(t_1 \dots t_n)) = \{t_n\}$, that is, $w(t_1 \dots t_n) \in W^{\{t_n\}}$. \square

Lemma 8.5. *Let (W, S) be an irreducible right-angled Coxeter system with $|\partial\Sigma(W, S)| > 2$. For each $w, v \in W$, there exists $x \in W$ such that*

- (1) $\ell(x) \leq 1$ and
- (2) $S(wx) \cup S(vx) \neq S$.

Proof. Let $w, v \in W$. If $S(w) \cup S(v) \neq S$ then $x = 1$ satisfies the conditions (1) and (2). We suppose that $S(w) \cup S(v) = S$.

Then $S(w) \cap S(v) = \emptyset$. Indeed if $s \in S(w) \cap S(v)$ then $st = ts$ for any $t \in S(w)$ by $s \in S(w)$ and Lemmas 6.1 and 8.2 (1), and also $st = ts$ for any $t \in S(v)$ by $s \in S(v)$ and Lemmas 6.1 and 8.2 (1). Hence $st = ts$ for any $t \in S(w) \cup S(v) = S$ and $W = W_{\{s\}} \times W_{S \setminus \{s\}}$ which contradicts to the assumption (W, S) is irreducible. Thus $S(w) \cap S(v) = \emptyset$.

Let $s_0 \in S(w)$. Since (W, S) is an irreducible right-angled Coxeter system, $o(s_0 t_0) = \infty$ for some $t_0 \in S$ by Lemma 8.2 (2). Then $t_0 \in S(v)$, because $W_{S(w)}$ and $W_{S(v)}$ are finite by Lemma 6.1 and $W_{\{s_0, t_0\}}$ is infinite. Here by Lemma 8.3,

$$S(vs_0) = \{s_0\} \cup \{t \in S(v) \mid ts_0 = s_0 t\}.$$

If $S(ws_0) \cup S(vs_0) \neq S$ then $x = s_0$ satisfies the conditions (1) and (2). We suppose that $S(ws_0) \cup S(vs_0) = S$. Since $t_0 \notin S(vs_0)$, $t_0 \in S(ws_0)$. Hence $t_0 s = st_0$ for each $s \in S(ws_0) = S \setminus S(vs_0)$. Here

$$\begin{aligned} S \setminus S(vs_0) &= S \setminus (\{s_0\} \cup \{t \in S(v) \mid ts_0 = s_0 t\}) \\ &\supset S \setminus (\{s_0\} \cup S(v)) \\ &= S(w) \setminus \{s_0\}, \end{aligned}$$

since $S(w) \cup S(v) = S$ and $S(w) \cap S(v) = \emptyset$. Hence $t_0 s = st_0$ for any $s \in S(w) \setminus \{s_0\}$. Since $t_0 \in S(v)$, we also have that $t_0 s = st_0$ for any $s \in S(v) = S \setminus S(w)$. Thus $t_0 s = st_0$ for any $s \in S \setminus \{s_0\}$. Here t_0 is an arbitrary element of S with $o(s_0 t_0) = \infty$.

Let $A = \{t \in S \mid o(s_0 t) = \infty\}$. Then $st = ts$ for any $t \in A$ and $s \in S \setminus \{s_0\}$ by the above argument. Also $s_0 t = ts_0$ for any $t \in S \setminus A$ by the definition of A , since (W, S) is right-angled. Hence we obtain that

$$W = W_{A \cup \{s_0\}} \times W_{S \setminus (A \cup \{s_0\})}.$$

Since (W, S) is irreducible, $S \setminus (A \cup \{s_0\}) = \emptyset$ and $S = A \cup \{s_0\}$. Hence

$$W = W_{\{s_0\}} * W_A.$$

Here s_0 is an arbitrary element of $S(w)$.

If there does not exist $x \in W$ which satisfies the conditions (1) and (2), then by the same argument for $t_0 \in S(v)$, we have that

$$W = W_{\{s_0\}} * W_{\{t_0\}} \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

Then the boundary $\partial\Sigma(W, S)$ is two-points set, which contradicts to the assumption $|\partial\Sigma(W, S)| > 2$.

Thus there exists $x \in W$ which satisfies the conditions (1) and (2). \square

Using Theorem 7.1 and lemmas above, we prove Theorem 8.1.

Proof of Theorem 8.1. Let (W, S) be an irreducible right-angled Coxeter system with $|\partial\Sigma(W, S)| > 2$. To show that the boundary $\partial\Sigma(W, S)$

is a scrambled set, by Theorem 7.1, we prove that there exist $s_0, t_0 \in S$ and a number $K > 0$ such that

- (1) $o(s_0 t_0) = \infty$ and
- (2) for each $w, v \in W$, there exists $x \in W$ such that $\ell(x) \leq K$ and $wx, vx \in W^{\{s_0\}}$.

Let $s_0 \in S$ and let $w, v \in W$. By Lemma 8.5, there exists $x_0 \in W$ such that $\ell(x_0) \leq 1$ and $S(wx_0) \cup S(vx_0) \neq S$. Then there is $t_1 \in S \setminus S(wx_0) \cup S(vx_0)$. Since (W, S) is an irreducible right-angled Coxeter system, by Lemma 8.2 (2), there exist $t_2, \dots, t_n \in S$ such that $o(t_i t_{i+1}) = \infty$ for each $i \in \{1, \dots, n-1\}$, $t_n = s_0$ and $\{t_1, t_2, \dots, t_n\} = S$.

Then by Lemma 8.4,

$$\begin{aligned} wx_0(t_1 t_2 \cdots t_n) &\in W^{\{t_n\}} = W^{\{s_0\}} \text{ and} \\ vx_0(t_1 t_2 \cdots t_n) &\in W^{\{t_n\}} = W^{\{s_0\}}. \end{aligned}$$

Hence we can take a large number $K > 0$ such that for each $w, v \in W$, there is $x \in W$ such that $\ell(x) \leq K$ and $wx, vx \in W^{\{s_0\}}$, because S is finite. Also there exists $t_0 \in S$ with $o(s_0 t_0) = \infty$ by Lemma 8.2 (2).

Therefore the boundary $\partial\Sigma(W, S)$ is a scrambled set by Theorem 7.1. \square

Let (W, S) be a Coxeter system. There exists a unique decomposition $\{S_1, \dots, S_r\}$ of S such that W is the direct product of the parabolic subgroups W_{S_1}, \dots, W_{S_r} and each Coxeter system (W_{S_i}, S_i) is irreducible ([1], [20, p.30]). We define

$$\tilde{S} := \bigcup \{S_i \mid W_{S_i} \text{ is infinite}\}.$$

The Coxeter system (W, S) determines the subset \tilde{S} of S . By the definition,

$$W = W_{\tilde{S}} \times W_{S \setminus \tilde{S}},$$

$W_{\tilde{S}}$ is infinite and $W_{S \setminus \tilde{S}}$ is finite. Also it is known that the parabolic subgroup $W_{\tilde{S}}$ is the minimum parabolic subgroup of finite index in W ([11, Corollary 3.3]).

By Theorem 8.1 and [18, Theorem 5.1], we obtain equivalent conditions of a right-angled Coxeter system whose boundary is a scrambled set.

Corollary 8.6. *Let (W, S) be a right-angled Coxeter system with $|\partial\Sigma(W, S)| > 2$. Then the following statements are equivalent:*

- (1) $\partial\Sigma(W, S)$ is a scrambled set.
- (2) $\partial\Sigma(W, S)$ is minimal.
- (3) $(W_{\tilde{S}}, \tilde{S})$ is irreducible.

Proof. (2) \Leftrightarrow (3): The statements (2) and (3) are equivalent by [18, Theorem 5.1].

(1) \Rightarrow (3): Suppose that $(W_{\tilde{S}}, \tilde{S})$ is *not* irreducible. Then $W_{\tilde{S}} = W_{S_1} \times W_{S_2}$ for some $S_1, S_2 \subset \tilde{S}$, where W_{S_1} and W_{S_2} are infinite by the definition of \tilde{S} . This implies that

$$\Sigma(W_{\tilde{S}}, \tilde{S}) = \Sigma(W_{S_1}, S_1) \times \Sigma(W_{S_2}, S_2)$$

and

$$\begin{aligned} \Sigma(W, S) &= \Sigma(W_{\tilde{S}}, \tilde{S}) \times \Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S}) \\ &= \Sigma(W_{S_1}, S_1) \times \Sigma(W_{S_2}, S_2) \times \Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S}). \end{aligned}$$

Here $\Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S})$ is bounded, because $W_{S \setminus \tilde{S}}$ is finite. Hence $\Sigma(W_{S_1}, S_1) \times \Sigma(W_{S_2}, S_2)$ is quasi-dense in $\Sigma(W, S)$. By Theorem 5.5, we obtain that $\partial\Sigma(W, S)$ is *not* a scrambled set.

(3) \Rightarrow (1): Suppose that $(W_{\tilde{S}}, \tilde{S})$ is irreducible. By Theorem 8.1, the boundary $\partial\Sigma(W_{\tilde{S}}, \tilde{S})$ is a scrambled set. Since

$$\Sigma(W, S) = \Sigma(W_{\tilde{S}}, \tilde{S}) \times \Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S})$$

and $\Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S})$ is bounded,

$$\partial\Sigma(W, S) = \partial\Sigma(W_{\tilde{S}}, \tilde{S}).$$

Therefore $\partial\Sigma(W, S)$ is a scrambled set. \square

9. REMARK

We can find some similarity between the conditions of CAT(0) groups and Coxeter groups whose boundaries are scrambled sets in this paper and the known conditions of CAT(0) groups and Coxeter groups whose boundaries are minimal sets in [12], [14], [16] and [18]. The relation is unknown now in general. The author has a question: Is it the case that a boundary of a CAT(0) group is a scrambled set if and only if it is minimal?

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